EXPLICIT CONNECTIONS WITH SU(2)-MONODROMY

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Abstract. The pure braid group Γ of a quadruply-punctured Riemann sphere acts on the $\mathrm{SL}(2,\mathbb{C})$-moduli $\mathcal{M}$ of the representation variety of such sphere. The points in $\mathcal{M}$ are classified into Γ-orbits. We show that, in this case, the monodromy groups of many explicit solutions to the Riemann-Hilbert problem are subgroups of SU(2). Most of these solutions are examples of representations that have dense images in SU(2), but with finite Γ-orbits in $\mathcal{M}$. These examples relate to explicit immersions of constant mean curvature surfaces.

1. Introduction

Let G be an algebraic Lie group with Lie algebra $\mathfrak{g}$. Denote by $\mathcal{O}_G$ and $\mathcal{O}_{\mathfrak{g}}$ the categorical quotients of the G-adjoint actions on G and $\mathfrak{g}$, respectively. The exponential map $\mathrm{E} : \mathfrak{g} \to G$ induces the diagonal exponential maps $\mathrm{E} : \mathcal{O}_{\mathfrak{g}} \to \mathcal{O}_G$ and $\mathcal{O}_{\mathfrak{g}}^n \to \mathcal{O}_G^n$.

Choose n and let

$$C = \{c \in \mathbb{P}^1(\mathbb{C})^n : c_n = \infty; c_i \neq c_j \text{ if } i \neq j\},$$

$$\mathcal{X}(\mathfrak{g}) = \{X \in \mathfrak{g}^n : \sum_{i=1}^n X_i = 0\}.$$

For each $c \in C$, let $\Sigma = \Sigma(c) = \mathbb{P}^1(\mathbb{C}) \setminus \{c_1, \cdots, c_n\}$. Fix a base point p and let $\pi_1 = \pi_1(\Sigma, p)$ be the fundamental group. The representation variety $\mathrm{Hom}(\pi_1, G)$ identifies with

$$\mathcal{R}(G) = \{A \in G^n : \prod_{i=1}^n A_i = e\}$$

which has a natural variety structure inherited from G. The diagonal G-adjoint actions on $\mathcal{X}(\mathfrak{g})$ and $\mathcal{R}(G)$ give categorical quotients $\mathcal{U}(G)$ and $\mathcal{M}(G)$, respectively. Denote by P the projections $\mathcal{R}(G) \to \mathcal{O}_G^n$ and $\mathcal{X}(\mathfrak{g}) \to \mathcal{O}_{\mathfrak{g}}^n$. We do not distinguish $X \in \mathcal{X}(\mathfrak{g})$ and $A \in \mathcal{R}(G)$ from their respective images in $\mathcal{U}(G)$ and $\mathcal{M}(G)$, the group is always

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assumed to be G and we shorten $\mathcal{R}(G)$ to $\mathcal{R}$ and etc, unless otherwise specified.

The pure braid group Γ of Σ acts on π_1, hence, on $\mathcal{R}$ and $\mathcal{M}$. The Γ-action preserves the fibre of P. For each $a \in \mathcal{O}_G^n$ (resp. $\theta \in \mathcal{O}_{\mathfrak{g}}^n$), let $\mathcal{R}_a = P^{-1}(a) \subset \mathcal{R}$ (resp. $\mathcal{X}_\theta = P^{-1}(\theta)$) and $\mathcal{M}_a = P^{-1}(a) \subset \mathcal{M}$ (resp. $\mathcal{U}_\theta = P^{-1}(\theta)$).

For $z \in \Sigma(c)$ and $X \in \mathcal{X}$,

$$D_X = \partial + \sum_{i=1}^{n-1} \frac{X_i}{z - c_i}$$

is a flat connection on Σ. Each D_X induces a representation $\pi_1 \to G$. This give rise to monodromy maps $\mathrm{hol} : \mathcal{X} \to \mathcal{R}$ and $\mathcal{U} \to \mathcal{M}$.

The Riemann-Hilbert problem concerns the surjectivity of hol. It is an existential question, but one may ask the constructive Riemann-Hilbert question: Given $A \in \mathcal{R}$, construct $X \in \mathcal{X}$ such that $\mathrm{hol}(X) = A$. When $n = 3$ and $G = \mathrm{SL}(2, \mathbb{C})$, there is *rigidity*, i.e. $\mathcal{M}_a$ consists of a point for a generic $a \in \mathcal{O}_G^3$. Notice that $\mathrm{E} \circ \mathrm{hol}$ equals E on $\mathcal{U}$ and the latter is simple to compute. If $\mathcal{M}_a$ is not empty, then, up to equivalence of representation, any $X \in \mathcal{X}$ with $\mathrm{E}(X) = a$ is a solution.

In general, $\mathcal{M}_a$ is a moduli with positive dimension for a generic a. The constructive Riemann-Hilbert problem involves solutions of non-linear differential equations and has been solved for the finite subgroups of $\mathrm{SU}(2)$ with $n = 4$ only recently [3, 4, 6]. Here we pose a related problem:

Question: Suppose $K < G$. Given $X \in \mathcal{X}$, determine whether, up to a G-inner isomorphism, $\mathrm{hol}(X)(\pi_1) < K$ or, equivalently, $\mathrm{hol}(X) \in \mathcal{R}(K)$.

This is an interesting problem in its own right, but also has applications. When $G = \mathrm{SL}(2, \mathbb{C})$ and $K = \mathrm{SU}(2)$, examples of such $X \in \mathcal{X}$ are related to explicit immersions of constant mean curvature surfaces into the Euclidean $\mathbb{E}^3$, the hyperbolic $\mathbb{H}^3$ and the standard S^3 [2, 11, 12, 13].

In a more general context, one may consider Σ to be a punctured Riemann surface of genus g. The mapping class group Γ fixing $\partial\Sigma$ acts on an analogous $\mathcal{M}_a(\mathrm{SL}(2, \mathbb{C}))$, preserving $\mathcal{M}_a(\mathrm{SU}(2))$. Suppose further that $g > 0$. If $\rho \in \mathcal{R}_a(\mathrm{SU}(2))$ and $\rho(\pi_1)$ is dense in $\mathrm{SU}(2)$, then the Γ-orbit is dense in $\mathcal{M}_a(\mathrm{SU}(2))$ [7, 8]. However this is no longer true when $g = 0$ (our present case) [9]. The results here also provide more such examples in the case of $g = 0$.

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2. Representation Varieties

From now on, assume $n = 4$ and $\mathrm{SU}(2) = K < G = \mathrm{SL}(2,\mathbb{C})$. For $H < G$, $A \in \mathcal{R}$ is said to be an H-class if, up to a G-inner isomorphism, $A(\pi_1) < H$. For a generic a, $\mathcal{M}_a$ is a smooth two dimensional moduli, hence, no longer rigid. However the points of $\mathcal{M}_a$ are classified into Γ-orbits. The finite orbits are actually subvarieties of $\mathcal{M}_a$ that are rigid in certain sense. As examples, if $A(\pi_1)$ is finite, then the Γ-orbit of A is finite. The classification of such A is elaborate, but carried out in [3, 4].

Up to a Möbius transformation, assume $c = \{0, 1, \infty, t\}$. The space $\mathcal{O}_G^4$ can be identified with $\mathbb{C}^4$ and the projection with the trace map [1, 5]

$$P(A_1, A_2, A_3, A_4) = (\mathrm{tr}(A_1), \mathrm{tr}(A_2), \mathrm{tr}(A_3), \mathrm{tr}(A_4)).$$

Let

$$a = (a_1, a_2, a_2, a_4) \in \mathcal{O}_G^4,$$
$$v = (v_1, v_2, v_3) = (\mathrm{tr}(A_1A_2), \mathrm{tr}(A_2A_3), \mathrm{tr}(A_1A_3)).$$

In this convention, the exponential map $\mathrm{E} : \mathcal{O}_{\mathfrak{g}} \to \mathcal{O}_G$ is defined by $\mathrm{E}(\theta) = 2\cos(\pi\theta)$. By [1, 5],

$$\mathcal{M} = \{(a, v) \in \mathbb{C}^7 : f(a, v) = 0\}, \quad \mathcal{M}_a = \{v \in \mathbb{C}^3 : f(a, v) = 0\},$$

where

$$\begin{aligned}(1)\quad f(a, v) = {}& v_1^2 + v_2^2 + v_3^2 + v_1v_2v_3 - (a_1a_2 + a_3a_4)v_1 - (a_1a_4 + a_2a_3)v_2 \\ & -(a_1a_3 + a_2a_4)v_3 + (a_1^2 + a_2^2 + a_2^2 + a_4^2 + a_1a_2a_3a_4 - 4).\end{aligned}$$

The group Γ has three generators τ_1, τ_2 and τ_3 with its actions on $\mathcal{M}_a$ [1, 5]:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \xmapsto{\tau_1} \begin{bmatrix} v_1 \\ a_1a_4 + a_2a_3 - v_1(a_1a_3 + a_2a_4 - v_1v_2 - v_3) - v_2 \\ a_1a_3 + a_2a_4 - v_1v_2 - v_3 \end{bmatrix},$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \xmapsto{\tau_2} \begin{bmatrix} a_1a_2 + a_3a_4 - v_2v_3 - v_1 \\ v_2 \\ a_2a_4 + a_1a_3 - v_2(a_1a_2 + a_3a_4 - v_2v_3 - v_1) - v_3 \end{bmatrix},$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \xmapsto{\tau_3} \begin{bmatrix} a_1a_2 + a_3a_4 - v_3(a_2a_3 + a_1a_4 - v_3v_1 - v_2) - v_1 \\ a_2a_3 + a_1a_4 - v_3v_1 - v_2 \\ v_3 \end{bmatrix}.$$

A representation $A \in \mathcal{R}$ is an SU(2)- or $\mathrm{SL}(2,\mathbb{R})$-class if and only if a and v are real [1]. Define interval

$$I_{s,t} = [\frac{st - \sqrt{(s^2-4)(t^2-4)}}{2}, \ \frac{st + \sqrt{(s^2-4)(t^2-4)}}{2}].$$

Then by [1],

Lemma 1. *A representation A is an* SU(2)*-representation if and only if $v \in \mathbb{R}^3 \subset \mathbb{C}^3$, $A \in \mathcal{M}_a$ with $a \in [-2,2]^4 \subset \mathbb{C}^4$ and $I_{a_1,a_2} \cap I_{a_3,a_4} \neq \emptyset$.*

Remark 2. *Let $a \in \mathcal{O}_G^4$. Suppose $\Delta < \Gamma$ has finite index. Let $\mathcal{M}_a^\Delta \subset \mathcal{M}_a$ be the Δ-invariant subvariety. The subspace $\mathcal{M}_a^\Delta$ is discrete in $\mathcal{M}_a$ and one may determine whether $\mathcal{M}_a^\Delta$ consists of* SU(2)*-classes by Lemma 1 (See* [9] *for an example).*

3. Flat connections

Let $\theta \in \mathcal{O}_{\mathfrak{g}}$ and $a = \mathrm{E}(\theta)$ which equals to $2\cos(\pi\theta)$ in our convention. Given $A \in \mathcal{R}_a$, explicit solutions of $X = (X_1, X_2, X_3, X_4) \in \mathcal{X}_\theta$ with $\mathrm{hol}(X) = A$ are related to solutions of the Painlevé VI equation [3, 4]:

$$\begin{aligned} \frac{d^2y}{dt^2} - \frac{1}{2}(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t})(\frac{dy}{dt})^2 + (\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t})\frac{dy}{dt} \\ = \frac{t(y-1)(y-t)}{t^2(t-1)^2}(r_1 + r_2\frac{t}{y^2} + r_3\frac{t-1}{(y-1)^2} + r_4\frac{t(t-1)}{(y-t)^2}), \end{aligned} \tag{2}$$

where

$$r_1 = \frac{(\theta_4-1)^2}{2}, \ r_2 = -\frac{\theta_1^2}{2}, \ r_3 = \frac{\theta^3}{2}, \ r_4 = \frac{(1-\theta_2^2)}{2}.$$

The group Γ acts on the moduli of flat connections, hence, on the solutions of the Painlevé VI equation [3, 4]. An algebraic solution to the Painlevé VI equation has a finite Γ-orbit (See [3, 4]) with isotropy subgroup $\Delta < \Gamma$. Once such a solution is found, one may deform it in many ways to obtain Δ-invariant families of solutions. For example, equation (2) is actually a family of equations parameterized by θ. Hence if y is an explicit solution, then y is a solution to a family of Painlevé VI equations if θ is deformed in such a way that the right hand side of Equation (2) remains constant. This family of equations then correspond to a θ-family of solutions $\Theta \subset \mathcal{X}^\Delta$. Set $\Theta_\theta = \Theta \cap \mathcal{X}_\theta$. It then follows that

Theorem 3. $\mathrm{hol}(\Theta_\theta) \subset \mathcal{M}_a^\Delta$. *Hence if $\mathcal{M}_a^\Delta$ consists of only* $\mathrm{SL}(2,\mathbb{R})$*- or* SU(2)*-classes (See Remark 2), then Θ_θ consists of only* $\mathrm{SL}(2,\mathbb{R})$*- or* SU(2)*-connections, respectively.*

4. Examples

Consider ([4] §3, Example 3). Let $A = \mathrm{hol}(X)$. Then $A(\pi_1) < K$ is contained in the symmetry group of the tetrahedron and the Γ-orbit of A consists of exactly two points. By a direct computation, Θ is parameterized by the affine variety

$$\{\theta \in \mathbb{C}^4 : -\theta_2^2 + \theta_3^2 = 0, 1 - \theta_1^2 - 2\theta_4 + \theta_4^2 = 0\}.$$

Moreover $A = (0, 1, 0) \in \mathcal{M}_{(1,-1,-1,-1)}$ and

$$\Delta = \langle \tau_2, \tau_1^2, \tau_3^2 \rangle. \tag{3}$$

By a direct computation, using Gröbner basis method, the subvariety $\mathcal{M}^\Delta \subset \mathcal{M}$ is defined by the ideal

$$(-4v_3+v_2^2v_3, -2v_1-v_2v_3, 4-2a_3^2-2a_4^2+a_3^2a_4^2+a_3^2v_2-a_4^2v_2-v_2^2, a_2-a_3, a_1+a_4)$$

Hence if $a = (a_1, a_2, a_2, -a_1)$, then

$$\mathcal{M}_a^\Delta = \{(0, 2 - a_1^2, 0), (0, a_2^2 - 2, 0)\}.$$

Furthermore, if $a \in [-2, 2]^4$ satisfies the additional hypothesis of Lemma 1, then $\mathcal{M}_a^\Delta$ consists of SU(2)-classes. Since the Γ-action on $\mathrm{hol}(S)$ satisfies the equations (3) for any $S \in \Theta$, $\mathrm{hol}(S) \subset \mathcal{M}^\Delta$.

Remark 4. *The* SU(2)*-classes in Theorem 3 are the ones found in* [9]*. We emphasize here that the matrices in* Θ *can be explicitly computed, but since the formulas are rather complicated, we refer to (*[4]*, Appendix A) for details.*

For the case of ([4] §3, Example 4), $a = (-2 + a_3^2, a_3, a_3, -1)$ and

$$\mathcal{M}_a^\Delta = \{(0, 1, 0), (0, 1, -3a_3 + a_3^3), (-3a_3 + a_3^3, 1, 0)\}.$$

For ([4] §3, Example 5), $a = (a_3, a_3, a_3, 0)$ and

$$\mathcal{M}_a^\Delta = \{(1, -2 + a_3^2, 1), (1, 1, 1), (-2 + a_3^2, 1, 1), (1, 1, -2 + a_3^2)\}.$$

Case ([4] §3, Example 6) is rigid.

One can similarly work out the octahedron cases [4] and the icosahedron cases [3].

5. Conclusions

To summarize the algorithm: For $[\Gamma : \Delta] < \infty$, let $\mathcal{M}^\Delta \subset \mathcal{M}$ be the Δ-invariant subvariety. Apply the methods in [3, 4, 6] to compute the family $\Theta \subset \mathcal{X}^\Delta$. Let $\theta \in P(\Theta)$ and $a = \mathrm{E}(\theta) \in \mathcal{O}_G^4$. The subspace $\mathcal{M}_a^\Delta$ is discrete in $\mathcal{M}_a$ and one may determine whether $\mathcal{M}_a^\Delta$ consists of $\mathrm{SL}(2, \mathbb{R})$- or SU(2)-classes by Lemma 1.

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